

## BMEN E4001x: Quantitative Physiology I / Molecular and Cellular Systems

### Notes 08 - Membrane Potentials and Electrodifusion

#### Nelson Chapter 7, Boal, K&S Chapter 2, B&B Chapter 6

Last time, we focused on the role of pore, channel, carrier, and pump proteins in moving compounds with or against a chemical gradient. Many of these compounds are charged ions, so voltages across the membrane will certainly play a part in all of this. For reference, here are the “standard” ion concentrations inside and outside a typical cell.

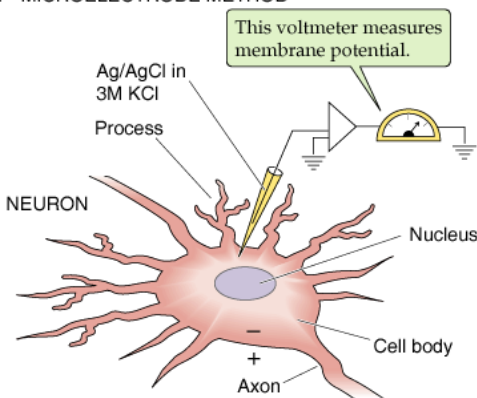
Ion	concentration (mM)	
	interstitial space	cell (“typical”)
Na <sup>+</sup>	145	15
K <sup>+</sup>	4.5	120
Ca <sup>2+</sup>	1.2	1 x 10 <sup>-7</sup>
Mg <sup>2+</sup>	.55	1
Cl <sup>-</sup>	116	20
HCO <sub>3</sub> <sup>-</sup>	25	15
glucose	5.9	low

At some point, you encountered the concept of transmembrane voltage; we’re going to look into this closer, and explore some of the complexities.

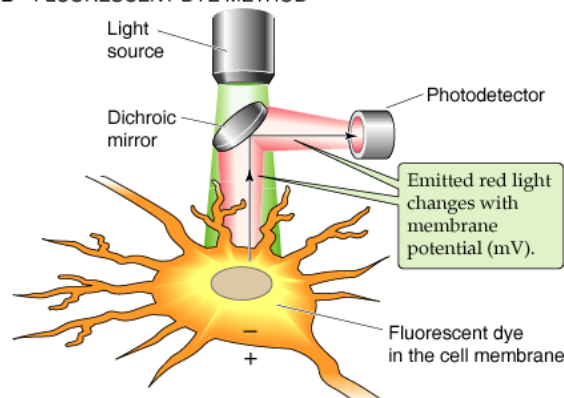
First of all, how do you measure it, where is it found?

Certainly, the central nervous system and neuron activity is a big instance of where these voltages come from. B&B, Chapter 6, goes over two main methods of measuring these:

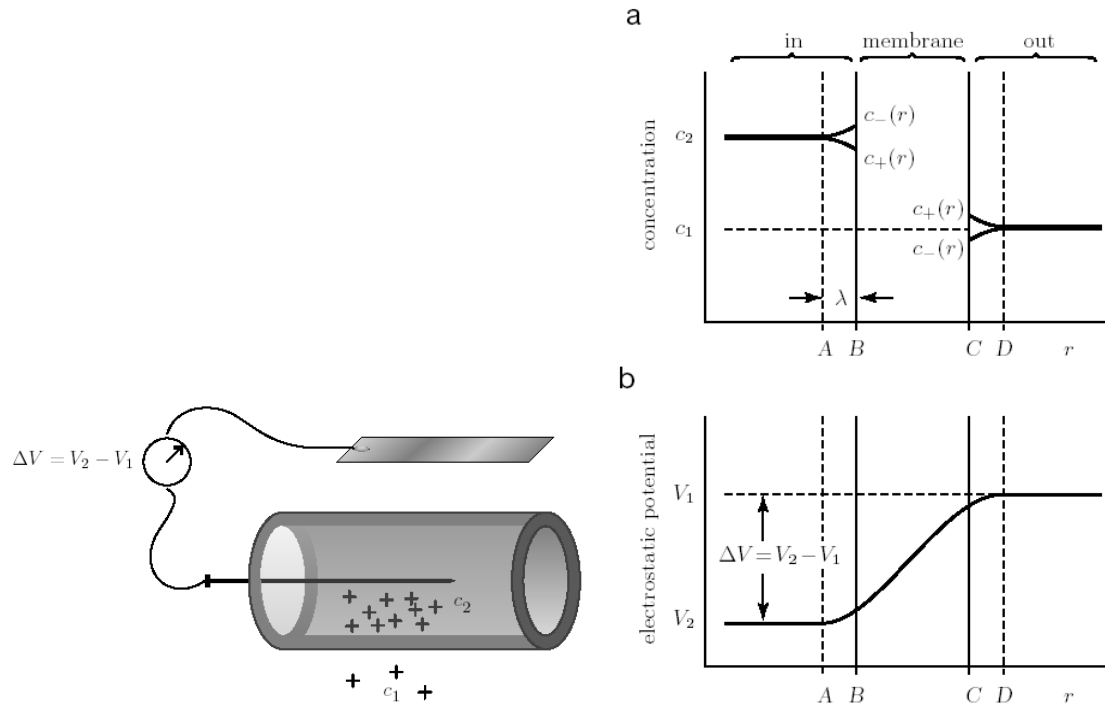
#### A MICROELECTRODE METHOD



#### B FLUORESCENT DYE METHOD



We’ll start here looking at where resting potentials come from, the foundation for more complex behaviors, and explore some of the spatial aspects of this system.



Start with KCl inside, no KCl outside, 0 volts across the membrane.

Note that voltage is read with respect to outside.

Membrane permeant to K, but not Cl; channels in action here.

Flow of K through the membrane carries positive charge outward, leaving the inside of the cell negative. Now, from the point of view of ions either inside or outside the cell, they see a barrier carrying a slight charge; from the outside, the cell is slightly negative, while from the inside, the barrier is slightly positive. As a result, there is a slight accumulation of K on the outside of the cell, and depletion inside the cell, opposite for Cl.

Note: there isn't a bulk unbalance of charges in solution; charge unbalance can be tolerated at small scales (thanks to thermal kicks), but cannot exist at bulk scales. For example, no solution of  $\text{Cl}^-$  without counterions.

## How “thick” is this charged layer?

### ***Charged plate in a solution with counterions:***

Assume surface charge of  $-\sigma_s$ . Mean-field theory, in which we can reduce this to 1D.

### **Poisson-Boltzmann equation:**

Gauss law

$$\int \mathbf{E} \cdot \hat{\mathbf{n}} \, dA = \frac{1}{\epsilon} \int \rho \, dV$$

$\epsilon$  = permittivity of medium;  $\epsilon_0$  = vacuum =  $8.85 \times 10^{-12} \text{ C}^2/(\text{N}\cdot\text{m}^2)$

$\rho$  = density of charge

In differential form,

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon} \rho$$

And taking in one dimension, to get the final form

$$E = -\nabla V = -\frac{dV}{dx}$$

$$\frac{d^2 V}{dx^2} = -\frac{1}{\epsilon} \rho$$

This is the Poisson equation, relating electric potential to charge distribution

Boltzmann equation, independent counterions

$$c_+(x) = c_0 \exp\left(-\frac{qV(x)}{k_B T}\right)$$

One leads to the other. So, must solve these simultaneously. This is the Poisson-Boltzmann equation:

$$\frac{d^2 V}{dx^2} = -\frac{1}{\epsilon} q c_0 \exp\left(-\frac{qV(x)}{k_B T}\right)$$

Solving (ala Boal, intermembrane forces)

$$\bar{V}(x) \equiv \frac{neV(x)}{k_B T}; q = ne$$

$$\frac{d^2 \bar{V}}{dx^2} = -4\pi l_B n^2 c_0 \exp(-\bar{V})$$

where  $l_B$  is the Bjerrum length

$$l_B \equiv \frac{e^2}{4\pi\epsilon k_B T}$$

one more substitution, to simplify

$$K^2 \equiv 2\pi l_B n^2 c_0$$

$$\frac{d^2 \bar{V}}{dx^2} = -2K^2 \exp(-\bar{V})$$

To solve, multiply both sides by  $d\bar{V}/dx$  then use the following relations:

$$\frac{d\bar{V}}{dx} \frac{d^2 \bar{V}}{dx^2} = \frac{1}{2} \frac{d\left[\left(\frac{d\bar{V}}{dx}\right)^2\right]}{dx}; \text{ from: } \frac{d f^n(x)}{dx} = n f^{n-1}(x) \frac{df(x)}{dx}$$

$$\exp(-\bar{V}) \frac{d\bar{V}}{dx} = -\frac{d(\exp(-\bar{V}))}{dx}$$

to rewrite the equation as:

$$\frac{d\left[\left(\frac{d\bar{V}}{dx}\right)^2\right]}{dx} = 4K^2 \frac{d(\exp(-\bar{V}))}{dx}$$

Integrate both sides by  $x$  and take a square root,

$$\frac{d\bar{V}}{dx} = 2K \exp\left(-\frac{\bar{V}}{2}\right)$$

NOTE: take the positive or negative square root as dictated by the species charge. Also, there's an integration constant, which was set to zero to get zero field gradient at infinity. Rewrite as

$$\exp\left(\frac{\bar{V}}{2}\right) d\bar{V} = 2K dx$$

Then integrate both sides as

$$\exp\left(\frac{\bar{V}}{2}\right) = K(x + \chi)$$

then, substituting back into the Boltzmann equation,

$$c_+(x) = c_0 (K(x + \chi))^{-2} = \frac{1}{2\pi l_B (x + \chi)^2}$$

Now, for neutrality, the integral of counterions must equal surface charge. Final result:

$$\chi = \frac{q = ne}{2\pi l_B \sigma_s} = \frac{2\epsilon k_B T}{ne\sigma_s}$$

So, this is the solution for counterions in solution.

### ***Charged plate in electrolyte***

In reality, there are salts, not just counterions around. This problem is much more complicated, but handled in much the same way as the above case. Poisson and Boltzmann forms are written for both positive and negative charges. An important result is:

$$\frac{d\bar{V}}{dx} = \pm 2 \sinh\left(\frac{\bar{V}}{2}\right) \frac{1}{l_D}; \quad l_D = (8\pi l_B c_s)^{-1/2};$$

$$l_B \equiv \frac{e^2}{4\pi\epsilon k_B T}; c_s = \text{farfield salt conc.}$$

which is then integrated.  $l_D$  is the Debye length. For small potentials, this can be simplified to the Debye-Huckel form:

$$\bar{V} = \bar{V}_0 \exp\left(-\frac{x}{l_D}\right)$$

So, the Debye length sets a fundamental distance for the influence of a charged surface in an electrolyte bath. For a typical Bjerrum length of 0.7 nm, 0.2M monovalent salt, body temp,  $l_D$  about 0.7 nm.

So, this described the thickness of the ion layer. This is a simplification that assumes a point charge; at these scales, the size of the ions makes a difference.

### **Nernst Potential**

From our discussion of sedimentation,

$$j(x) = -D \frac{dC(x)}{dx} + vC(x) \equiv 0$$

Solution is:  $C(x) = C_0 \exp(v \cdot x / D)$ .

In this case,  $v = \mu \cdot F = \mu \cdot q \cdot \varepsilon$ , where  $\varepsilon$  = electric field and  $q$  is charge on the diffusing species.

So,  $C(x) = C_0 \exp(q \cdot \varepsilon \cdot x / (k_B T))$

Now, taking  $C_1$  and  $C_2$  as the voltages taken at distances  $L$ , and  $\varepsilon = \Delta V / L$

$$C_2 = C_1 \cdot \exp(q \cdot (\Delta V / L) \cdot L / (k_B T))$$

This leads us to

$$\Delta V = V_2 - V_1 = V^{\text{Nernst}} \equiv -\frac{k_B T}{q} \ln\left(\frac{C_2}{C_1}\right)$$

Ion	concentration (mM)		
	interstitial space	cell ("typical")	$V^{\text{Nernst}}$ (mV) (37°C)
$\text{Na}^+$ ,	145	15	+61
$\text{K}^+$ ,	4.5	120	-85
$\text{Cl}^-$ ,	116	20	-47

Okay, relate back to the membrane. If we assume:

Field  $\varepsilon$  is present and uniform in the range of the lipid bilayer, and only there

The field is produced by the unbalance of ion charges, but the number of charges needed to create this potential doesn't significantly change bulk concentrations of soluble ions

The diffusing particles act independently, and don't alter the electric field

This is the potential needed to produce no net flux of ions across a region separating two solutions of different bulk concentrations of an ion. This is the Nernst potential.

For our example  $\text{K}^+$  ions selectively diffusing from a cell, the potential is negative, drawing the positive charges back towards the membrane. If the membrane potential is not as negative,  $\text{K}^+$  ions continue to flow out, decreasing the membrane potential further. If the membrane potential is more negative than the Nernst potential for  $\text{K}^+$ , these ions are, in effect, pulled back into the cell.

## Electrodiffusion (Goldman-Hodgkin-Katz equations):

How does the presence of the electric field affect ion movement?

Start with the 1-D version of the Nernst-Planck equation we encountered earlier, slightly modified into  $x$ , and regrouped with respect to  $D$ .

$$j(x) = -D \frac{dC(x)}{dx} + D \frac{q\varepsilon}{k_B T} C(x); \text{ Nernst-Planck equation}$$

$q$  = charge on diffusing particle,  $\varepsilon$  = electric field;  $D = \mu k_B T$ ;  $\mu$  = mobility =  $v/f$

Rearranging this, we can get to:

$$\frac{dC(x)}{dx} - \frac{q\varepsilon}{k_B T} C(x) + \frac{j}{D} = 0$$

This can be solved, using the boundary conditions:  $C(0)=C_i$  and  $C(L)=C_e$

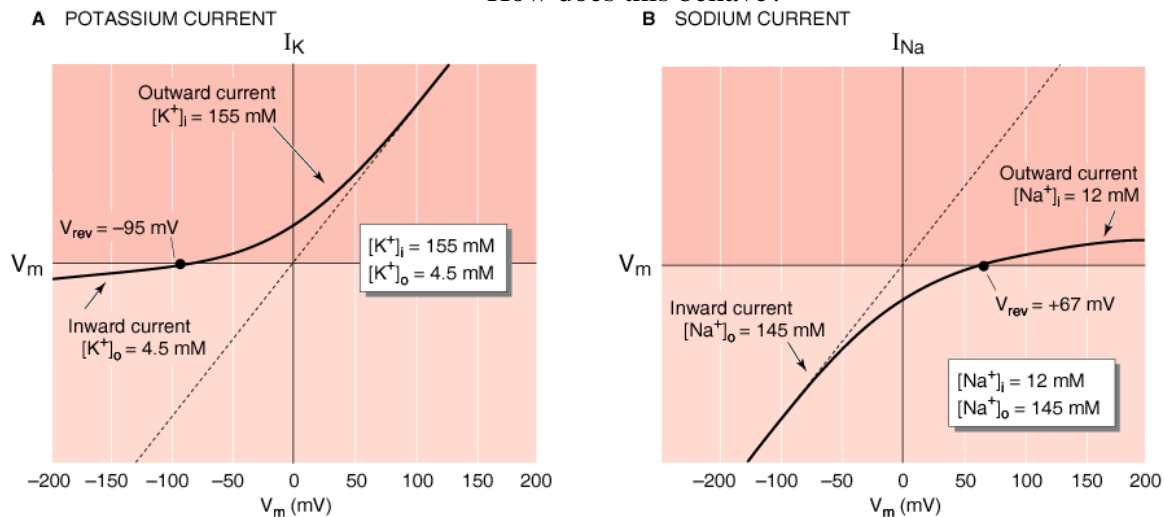
$$j = \frac{D}{L} \frac{q\epsilon L}{k_B T} \frac{C_i - C_e \exp\left(-\frac{q\epsilon L}{k_B T}\right)}{1 - \exp\left(-\frac{q\epsilon L}{k_B T}\right)}$$

Since  $j$  is in molar flux, to get current per area, multiply by charge per mole,  $q^*N_A$ .

$$I = \frac{D}{L} \frac{q^2 N_A \epsilon L}{k_B T} \frac{C_i - C_e \exp\left(-\frac{q\epsilon L}{k_B T}\right)}{1 - \exp\left(-\frac{q\epsilon L}{k_B T}\right)} = P \frac{q^2 N_A \Delta V}{k_B T} \frac{C_i - C_e \exp\left(-\frac{q\Delta V}{k_B T}\right)}{1 - \exp\left(-\frac{q\Delta V}{k_B T}\right)}; P = \frac{D}{L} \text{ or } \frac{\beta D}{L} \text{ (B \& B)}$$

THIS IS THE GOLDMAN-HODGKIN-KATZ equation.

How does this behave?



- Starting with  $K^+$ . Recall that in our system, voltage started at zero, and got more negative as  $K^+$  flowed out. For membrane voltages above  $V_K^{Nernst}$ , chemical and electrical forces work in the same direction, current versus voltage is somewhat linear and approaches a line that passes through zero.
- For our situation, as the membrane voltage became more negative than the Nernst potential, now electrical and chemical forces counteract, imparting a deviation from the linear line. Note that at a certain voltage, the Nernst potential, flux is zero.
- Similarly for sodium, when chemical and electric forces are in the same direction, approaches linear correlation. Deviations when electric and chemical forces oppose.
- This solution assumes a homogenous slab, independent diffusion, uniform electric field. Thus, this good for simple diffusion as presented. You can imagine migrating this to the case with a more structured diffusion, as we ended up with for the porous membrane.